Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 254 (2008) 1269–1281

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Measures on double or resonant eigenvalues for linear Schrödinger operator

Didier Bresch [a,*], Benoît Desjardins [b,c], Emmanuel Grenier [d]

[a] *LAMA, UMR CNRS 5127, Université de Savoie, 73376 Le Bourget du Lac cedex, France*
[b] *Modélisation Mesure et Applications S.A., 66 avenue des Champs Elysées, 75008 Paris, France*
[c] *ENS Ulm, DMA, 45 rue d'Ulm, 75230 Paris cedex 05, France*
[d] *U.M.P.A., École Normale Supérieure de Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France*

Received 1 May 2007; accepted 1 November 2007

Available online 14 January 2008

Communicated by J. Bourgain

Abstract

In this paper we consider linear Schrödinger operator with double or resonant eigenvalues. The main result is the bound of the measure (in a suitable space of functions) of the potentials leading to such double or resonant eigenvalues. Namely we present measure type estimates evaluating neighborhoods of the so-called double or resonant set.

Keywords: Linear Schrödinger operator; Measure type estimates; Multiple eigenvalues; Resonant set

1. Introduction

Let us consider the following linear Schrödinger operator

$$\mathcal{L}(V) = -\Delta + V$$

in a periodic setting ($x \in \mathbb{T}^d$ with $d \geqslant 1$). As $\mathcal{L}$ is a symmetric operator, with a compact inverse, there eigenvectors $\psi_j(V)$ form an Hilbert basis. Let $\lambda_j(V)$ be the corresponding (real) eigenvalues sorted in increasing order.

* Corresponding author.
E-mail addresses: didier.bresch@univ-savoie.fr (D. Bresch), benoit.desjardins@ens.fr (B. Desjardins), egrenier@umpa.ens-lyon.fr (E. Grenier).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.11.009

For some potentials $V(x)$ all the eigenvalues are simple, however for some potentials, multiple eigenvalues may occur, see for instance [12], [9]. In this paper we want to bound the measure (in a suitable space of functions) of the potentials leading to double or resonant eigenvalues. For reference books about resonance problems for Schrödinger operator, the reader is for example referred to [10], [9] and [11]. For other resonance studies, see also [6–8,16]. For considerations around existence or absence of possible eigenvalues for the Schrödinger operator, the reader is referred to [14,15]. See also [3] for supraconductivity applications and reference cited therein. Note that many people are interesting in the control of the resonances of such operators in dimension $d > 1$ in particular in view of applications to KAM theory (see for instance [1,2,5]).

Let B_s denote the usual Besov spaces $B_2^{s,\infty}$ and introduce

$$\Sigma_{j,k} = \{V \in B_s \mid \lambda_j(V) = \lambda_k(V)\}$$

together with, for $\varepsilon > 0$,

$$\Sigma_{j,k}^{\varepsilon} = \{V \in B_s \mid d_{B_s}(V, \Sigma_{j,k}) < \varepsilon\}.$$

Note that $\mathcal{L}(V)$ being symmetric, we expect $\Sigma_{j,k}$ to be of codimension 2 in B_s. The aim of this paper is to give a measure approach of this assertion, namely to bound the measure of $\Sigma_{j,k}^{\varepsilon}$ by $C\varepsilon^2$ for some constant C.

In fact we will be a little more precise and bound the measure of finite dimension approximations of $\Sigma_{j,k}$ and $\Sigma_{j,k}^{\varepsilon}$. For this, let P_N denote the projector on the N first Fourier components. Let

$$\Sigma_{j,k}^{N} = \{V = P_N V \mid \lambda_j(V) = \lambda_k(V)\}$$

and for $\varepsilon > 0$,

$$\Sigma_{j,k}^{N,\varepsilon} = \{V = P_N V \mid d_{B_s}(V, \Sigma_{j,k}^{N}) < \varepsilon\}.$$

In this paper we bound the measure of $\Sigma_{j,k}^{N,\varepsilon}$ uniformly in $\varepsilon < 1$ and N.

Of course we have to precise the types of measures we consider. We will focus on the following measures on Besov spaces $B_s = B_2^{s,\infty}$. We believe our proofs may be extended to many other cases, provided the measure is a tensor product of one-dimensional measures on each harmonic of V.

Let $(e_n)_{n\in\mathbb{N}}$ be a Fourier basis. Let μ be the Lebesgue measure on $\mathbb{R}$. We recall that if

$$V = \sum_n \alpha_n e_n,$$

its $B_2^{s,\infty}$ Besov norm is defined by

$$\|V\|_{B_s} = \sup_n |\alpha_n|\bigl(1 + n^s\bigr).$$

We define $\mu_{s,R}$ by

$$\mu_{s,R} = \bigotimes_{n\geqslant 0} \bigl(1 + n^s\bigr)\frac{\mu(\alpha_n)}{R}.$$

Note that if $B_s(R)$ denotes the ball of radius R in B_s,

$$\mu_{s,R}\big(B_s(R)\big) = 1.$$

So we want to prove that

$$\mu_{s,R}\big(\Sigma_{j,k}^{N,\varepsilon} \cap B_s(R)\big) \leqslant C\varepsilon^2 \tag{1}$$

uniformly in ε and N. More precisely we prove

Theorem 1.1. *Quasi-double eigenvalues for finite approximations. Let $s \in \mathbb{R}$ being large enough. Let $\rho_0 \in B^s$ and $R > 0$ be such that for $\rho \in \rho_0 + B_s(R)$, $\lambda_j(\rho)$ is never a triple eigenvalue. Let us also assume that when $\lambda_j(\rho)$ is a double eigenvalue ($\lambda_j(\rho) = \lambda_k(\rho)$), with an orthonormal basis of eigenvectors ($\psi_j(\rho), \psi_k(\rho)$), then the following non-degeneracy condition holds true*:

$$\mathrm{Vect}\big(|\psi_j|^2 - |\psi_k|^2, \psi_j\psi_k\big) \quad \text{is of dimension } 2.$$

Then there exists a constant C and an integer N_0 such that for every $0 < \varepsilon < 1$ and every $N > N_0$

$$\mu_{s,R,N}\big(\Sigma_{j,k}^{N,\varepsilon} \cap B_s(R)\big) \leqslant C\varepsilon^2,$$

and

$$\mu_{s,R}\big(\Sigma_{j,k}^{\varepsilon} \cap B_s(R)\big) \leqslant C\varepsilon^2.$$

Remark. Remark that we do not investigate uniformity in j in the previous theorem nor in Theorem 1.2 even if it could be a useful result to avoid all the resonances. Remark also that triple eigenvalues may be studied using similar methods even if it is more technical.

We will also detail a similar result on resonant sets. Let

$$\Sigma_{j,k,l} = \big\{V \in B_s \;\big|\; \lambda_j(V) + \lambda_k(V) = \lambda_l(V)\big\},$$
$$\Sigma_{j,k,l}^{\varepsilon} = \big\{V \in B_s \;\big|\; \big|\lambda_j(V) + \lambda_k(V) - \lambda_l(V)\big| < \varepsilon\big\}$$

and their finite dimension approximations

$$\Sigma_{j,k,l}^{N} = \big\{V = P_N V \;\big|\; \lambda_j(V) + \lambda_k(V) = \lambda_l(V)\big\},$$
$$\Sigma_{j,k,l}^{N,\varepsilon} = \big\{V = P_N V \;\big|\; \big|\lambda_j(V) + \lambda_k(V) - \lambda_l(V)\big| < \varepsilon\big\}.$$

We want to prove that

$$\mu_{s,R}\big(\Sigma_{j,k,l}^{N,\varepsilon} \cap B_s(R)\big) \leqslant C\varepsilon \tag{2}$$

uniformly in ε and N. More precisely we will prove

Theorem 1.2. *Quasiresonance for finite approximations. Let $s \in \mathbb{R}$ being large enough. Let $R > 0$ and $\rho_0 \in B^s(\Omega)$ such that for every $\rho \in \rho_0 + B_s(R)$, the eigenvalues $\lambda_j(\rho)$, $\lambda_k(\rho)$ and $\lambda_\ell(\rho)$ are simple, with related eigenvectors $\psi_j(\rho)$, $\psi_k(\rho)$ and $\psi_\ell(\rho)$. Let us assume moreover that*

$$\rho \mapsto \left|\psi_j(\rho)\right|^2 + \left|\psi_k(\rho)\right|^2 - \left|\psi_\ell(\rho)\right|^2 \quad \textit{never identically vanishes.}$$

Then there exists a constant $C_1 > 0$ and an integer N_0 such that for every $0 < \varepsilon < 1$ and every $N > N_0$

$$\mu_{s,R}\left(\Sigma_{j,k,\ell}^{N,\varepsilon}\right) \leqslant C_1 \varepsilon,$$

and such that for every $0 < \varepsilon < 1$,

$$\mu_{s,R}\left(\Sigma_{j,k,\ell}^{\varepsilon}\right) \leqslant C_1 \varepsilon.$$

The proof of these theorems has two steps. First in Section 2 we study the local geometry of the eigenvalues and eigenvectors, both in the simple and multiple case. Then in Section 3 we state a general result which enables to bound measures of neighborhoods starting from almost normal vector fields. Application of Section 3 to Section 2 ends the proof.

2. Geometry of eigenvalues and eigenvectors

2.1. Simple eigenvalues

Let us begin by a formal computation. Let λ_j and ψ_j be an eigenvalue and an eigenvector for the potential V. Let $\tilde{V}$ be a small perturbation. See for instance [4,13] for eigenvalues variation for Sturm–Liouville operators. Introducing a small parameter ε to better see the various orders of approximations, and assuming the eigenvalue and eigenvector of $V + \varepsilon\tilde{V}$ are of the form $\lambda_j + \varepsilon\tilde{\lambda}$ and $\psi_j + \varepsilon\tilde{\psi}$ up to higher order terms, we are lead to equal ε terms in

$$(-\Delta + V + \varepsilon\tilde{V})(\psi_j + \varepsilon\tilde{\psi}) = (\lambda_j + \varepsilon\tilde{\lambda})(\psi_j + \varepsilon\tilde{\psi})$$

which gives

$$\tilde{V}\psi_j + (-\Delta + V)\tilde{\psi} = \tilde{\lambda}\psi_j + \lambda_j\tilde{\psi}. \tag{3}$$

Projecting on ψ_j and using the equation satisfied by ψ_j, we get

$$d\lambda_j(V).\tilde{V} = \int \tilde{V}|\psi_j|^2 \tag{4}$$

and projecting on the orthogonal of ψ_j gives

$$d\psi_j(V).\tilde{V} = (-\Delta + V - \lambda_j)^{-1}\big(1 - \Pi_j(V)\big)\left(\int \tilde{V}|\psi_j|^2\psi_j - \tilde{V}\psi_j\right) \tag{5}$$

where $\Pi_j(V)$ denotes the orthogonal projection on ψ_j. At this point, (4) and (5) are purely formal and will be proved below.

2.2. Multiple eigenvalues

Let us go on with formal computations, assuming now that $\lambda_j(V)$ has multiplicity $d' \geqslant 1$. To simplify the notations assume, up to a reordering,

$$\lambda_1(V) = \lambda_2(V) = \cdots = \lambda_{d'}(V).$$

Let us look for approximate eigenvalues of the form

$$\phi_j = \psi_j + \varepsilon\tilde{\phi}_j$$

and assume that the projection of $\tilde{\phi}_j$ over $\mathrm{Vect}(\psi_1, \ldots, \psi_{d'})$ vanishes. Then

$$\mathcal{L}(V + \varepsilon\tilde{V})\phi_j = \lambda_j\psi_j + \varepsilon\tilde{V}\psi_j + \varepsilon\mathcal{L}(V)\tilde{\phi}_j + \varepsilon^2\tilde{V}\tilde{\phi}_j.$$

The components of $\mathcal{L}(V + \varepsilon\tilde{V})\phi_j$, up to terms of order ε^2 are

$$\lambda_j + \varepsilon\int \tilde{V}|\psi_j|^2$$

on ψ_j and

$$\varepsilon\int \tilde{V}\psi_j\bar{\psi}_k$$

on ψ_k with $k \neq j$, $1 \leqslant k \leqslant d'$. Let

$$\lambda_{\mathrm{lin},j,k} = \int \tilde{V}\psi_j\bar{\psi}_k.$$

We get

$$\begin{aligned}
&\mathcal{L}(V + \varepsilon\tilde{V})\phi_j - \lambda_j\phi_j - \varepsilon\sum_{k=1}^{d'}\lambda_{\mathrm{lin},j,k}\phi_k \\
&\quad = -\varepsilon\lambda_j\tilde{\phi}_j + \varepsilon\tilde{V}\psi_j + \varepsilon\mathcal{L}(V)\tilde{\phi}_j + \varepsilon^2\tilde{V}\tilde{\phi}_j - \varepsilon\sum_{k=1}^{d'}\lambda_{\mathrm{lin},j,k}\phi_k.
\end{aligned}$$

Note that by definition of $\lambda_{\mathrm{lin},j,k}$ the components of the right-hand side on ψ_k with $1 \leqslant k \leqslant d'$ are of order ε^2. And its component in the orthogonal space to $\mathrm{Vect}(\psi_1, \ldots, \psi_{d'})$ is of order ε^2 provided we choose

$$\tilde{\phi}_j = -\big(\mathcal{L}(V) - \lambda_j\big)^{-1}(1 - \Pi)\tilde{V}\psi_j$$

where Π is the orthogonal projector on $\mathrm{Vect}(\psi_1, \ldots, \psi_{d'})$.

Let A be defined in the basis $(\phi_k)_{1\leqslant k\leqslant d'}$ by

$$A = (\lambda_k \delta_{j,k} + \varepsilon \lambda_{\mathrm{lin},j,k})_{1\leqslant j,k\leqslant d'}.$$

Note that

$$\mathcal{L}(V + \varepsilon \tilde{V})\phi_j - A\phi_j$$

is of order ε^2, hence up to terms of order ε^2, $\mathcal{L}(V + \varepsilon \tilde{V})$ as the same eigenvalues as the $d' \times d'$ array

$$A = \left(\lambda_k \delta_{j,k} + \varepsilon \int \tilde{V} \psi_j \bar{\psi}_k\right)_{1\leqslant j,k\leqslant d'}.$$

To justify all the above assertions, note that provided s is large enough, $B_{s/2}$ norms of $\tilde{\phi}_j$ may be bounded in terms of B_s norms of V, $\tilde{V}$ and ψ_j. Similarly $B_{s/2}$ norms of $\mathcal{L}(V + \varepsilon \tilde{V})\phi_j - A\phi_j$ may be bounded by $C\varepsilon^2$, the constant depending on B_s norms of V.

As A is symmetric, there exist d' vectors

$$\theta_k = \sum_{j=1}^{d'} \lambda_{j,k}\phi_j$$

and d' scalars μ_k such that

$$A\theta_k = \mu_k \theta_k.$$

Note that $\mathcal{L}(V + \varepsilon \tilde{V})\theta_k - \mu_k\theta_k$ is of order ε^2 in $B_{s/2}$ norm. As $\mathcal{L}(V + \varepsilon \tilde{V})$ is diagonal in some orthonormal basis, this means that the various μ_k are close to the various $\lambda_j(V + \varepsilon \tilde{V})$ up to ε^2 terms, and similarly for the eigenvectors.

Let us now detail the cases $d' = 1$ and $d' = 2$. For $d' = 1$, A is simply a scalar and we get

$$\lambda_1(V + \varepsilon \tilde{V}) = \lambda_1(V) + \varepsilon \int \tilde{V} \psi_1 \bar{\psi}_1 + O\big(\varepsilon^2\big).$$

In particular λ_1 is differentiable and

$$d\lambda_1(V).\tilde{V} = \int \tilde{V} \psi_1 \bar{\psi}_1.$$

Moreover

$$\psi_1(V + \varepsilon \tilde{V}) = \psi_1(V) - \big(\mathcal{L}(V) - \lambda_j\big)^{-1}(1 - \Pi)\tilde{V}\psi_j + O\big(\varepsilon^2\big).$$

Hence the eigenvector ψ_1 is differentiable in V with

$$d\psi_1(V)\tilde{V} = -\big(\mathcal{L}(V) - \lambda_j\big)^{-1}(1 - \Pi)\tilde{V}\psi_1.$$

Let us turn to the case $d' = 2$. We get that the two eigenvalues $\lambda_1(V + \varepsilon\tilde{V})$ and $\lambda_2(V + \varepsilon\tilde{V})$ are given by

$$\lambda_1(V) + \frac{1}{2}\left(\int \tilde{V}|\psi_1|^2 + \int \tilde{V}|\psi_2|^2\right)$$
$$\pm \frac{1}{2}\sqrt{\left(\int \tilde{V}|\psi_1|^2 - \int \tilde{V}|\psi_2|^2\right)^2 + 4\left(\int \tilde{V}\psi_1.\bar{\psi}_2\right)^2} + O\big(\varepsilon^2\big).$$

Note that eigenvectors of $\mathcal{L}(V + \varepsilon\tilde{V})$ lie, up to $O(\varepsilon^2)$ terms in

$$\mathrm{Vect}(\psi_1, \psi_2) - \big(\mathcal{L}(V) - \lambda_j\big)^{-1}(1 - \Pi)\tilde{V}\,\mathrm{Vect}(\psi_1, \psi_2).$$

In particular $\mathrm{Vect}(\psi_1, \psi_2)$ changes smoothly with V.

We will define the plane Π by

$$\Pi(V) = \mathrm{Vect}\big(|\psi_1|^2 - |\psi_2|^2, \psi_1\psi_2\big).$$

Simple calculations show that $\Pi(V)$ is independent on the particular choice of the orthonormal basis of eigenvectors, and only depends on $\mathrm{Vect}(\psi_1, \psi_2)$. As a consequence, $\Pi(V)$ changes smoothly with V.

3. Vector spaces almost normal to a set

3.1. Definition

Let ϕ be some Lipschitz continuous function from B_s to $\mathbb{R}$ and from B_σ to $\mathbb{R}$. Let

$$\Sigma = \big\{x \bigm| \phi(x) = 0\big\} \subset B_s,$$

and let

$$\Sigma^N = \Sigma \cap \{x = P_N x\} \subset \mathbb{R}^N.$$

We also define for every $\varepsilon > 0$

$$\Sigma^\varepsilon = \big\{x \bigm| d_{B_s}(x, \Sigma) < \varepsilon\big\},$$

and

$$\Sigma^{N,\varepsilon} = \big\{x \bigm| x = P_N x,\ d_{B_s}\big(x, \Sigma^N\big) < \varepsilon\big\}.$$

We will play with three vector spaces: L^2 (space where the vector will take their values), B_σ (space where various error terms will be bounded), and B_s (extra regularity). We will assume

$$d < \sigma < s.$$

In particular $B_\sigma \subset L^2$.

Definition 3.1. Almost normals to Σ. A collection of d' smooth vector fields $u_1(x), \dots, u_{d'}(x)$ is called almost normal to Σ if

(H1) for every $1 \leqslant j \leqslant d'$, u_j is Lipschitz continuous from B_σ to L^2, with Lipschitz constant k_0 and continuous from B_s to B_σ;

(H2) there exists a constant $\gamma > 0$ such that for every $x \in B_\sigma$,

$$\left|\det\left(\left(u_j(x) \mid u_k(x)\right)_{L^2}\right)_{1 \leqslant j,k \leqslant d'}\right| > \gamma;$$

(H3) there exists smooth functions $\alpha_1(x), \dots, \alpha_{d'}(x)$, a constant $\alpha > 0$ and a constant $C_0 > 0$ such that for every x with $d(x, \Sigma^N) < \alpha$

$$\left|\left|\phi(y)\right| - \sqrt{\sum_{j=1}^{d'}\left(u_j(x).(y-x) - \alpha_j(x)\right)^2}\right| \leqslant C_0 \|y - x\|_{H^\sigma}^2$$

for every $y \in H^s$.

3.2. Measure of neighborhoods

Proposition 3.2. *Measure estimates. Let $s \in \mathbb{R}$ being large enough and $R > 0$. If there exists a collection of d' smooth vector fields $u_1(x), \dots, u_{d'}(x)$ which is almost normal to Σ then there exists a constant $C(R)$ such that for every ε small enough*

$$\mu_{s,R}\left(\Sigma^{N,\varepsilon} \cap B_s(R)\right) \leqslant C(R)\varepsilon^{d'} \tag{6}$$

and

$$\mu_{s,R}\left(\Sigma^{\varepsilon} \cap B_s(R)\right) \leqslant C(R)\varepsilon^{d'}. \tag{7}$$

Proof. Let $\eta > 0$. Let

$$M_\sigma = \sup_{1 \leqslant j \leqslant d} \sup_{x \in B_s(R)} \left\|u_j(x)\right\|_{B_\sigma}$$

which is finite using (H1). Note that for every j, provided σ is large enough,

$$\left\|u_j(x) - P_N u_j(x)\right\|_{L^2} \leqslant \frac{M_\sigma}{N},$$

therefore if we choose $N > N_0$ with

$$N_0 > \frac{M_\sigma}{\eta},$$

we get

$$\left\|u_j(x) - P_N u_j(x)\right\|_{L^2} < \eta.$$

Next we cut off high frequencies of x. If $P_N x = P_N y$ then using (H1)

$$\left\|u_j(x) - u_j(y)\right\|_{L^2} \leqslant k_0 \|x - y\|_{B_\sigma} \leqslant \frac{k_0}{N^{s-\sigma}} \|x - y\|_{B_s} \leqslant \frac{2Rk_0}{N^{s-\sigma}}.$$

We therefore choose $N \geqslant N_1$ such that

$$\frac{2k_0 R}{N_1^{s-\sigma}} \leqslant \eta.$$

Then for every $x \in B_s(R)$ and for every $N > N_1$,

$$\left\|u_j(x) - u_j\big(P_N(x)\big)\right\|_{L^2} \leqslant \eta. \tag{8}$$

Let

$$\eta' < \frac{\eta}{k_0},$$

where k_0 is the Lipschitz constant given by (H1). As $P_{N_1} B_s(R)$ is compact it may be covered by a finite number of balls $B_s(x_i, \eta')$ of radius η'. Now if x and y are such that $P_{N_1}(x)$ and $P_{N_1}(y)$ both are in $B_s(x_i, \eta')$, then

$$\left\|u_j(x) - u_j(y)\right\|_{L^2} \leqslant 3\eta.$$

Let

$$B_i^N = \big(B_s(x_i, \eta') \times \mathbb{R}^{N-N_1}\big) \cap B_s(R).$$

Let us fix some i and introduce

$$V_i = \mathrm{Vect}\big(P_{N_1} u_1(x_i), \ldots, P_{N_1} u_{d'}(x_i)\big).$$

For $x \in B_i^N$ we define

$$A_x = (x + V_i) \cap B_i^N.$$

Let

$$\theta_x = \min_{x' \in A_x} \left|\phi(x')\right|.$$

Let K be a Lipschitz constant for ϕ. Assume first $\theta_x > K\varepsilon$. If $x_0 \in \Sigma$ and $x' \in A_x$ then

$$\left|\phi(x') - \phi(x_0)\right| = \left|\phi(x')\right| \leqslant K \|x' - x_0\|_{H^s}$$

therefore

$$d(x', \Sigma) > \frac{\theta_x}{K} > \varepsilon,$$

which implies that $x' \notin \Sigma^{N,\varepsilon}$. Hence $A_x \cap \Sigma^{N,\varepsilon} = \emptyset$.

Let us now assume on the contrary $\theta_x \leqslant K\varepsilon$. Let x' be a point where θ_x is attained. Assumption (H3) with $x = x'$ and $y = x'$ gives

$$\sqrt{\sum_{1\leqslant j\leqslant d} \left|\alpha_j(x')\right|^2} = \left|\phi(x')\right| \leqslant K\varepsilon.$$

We also have

$$u_j(x').y = P_{N_1}u_j(x_i).y + \big(u_j(x') - u_j(x_i)\big).y + \big(u_j(x_i) - P_{N_1}u_j(x_i)\big).y$$

and

$$\left\|u_j(x') - u_j(x_i)\right\|_{L^2} \leqslant 3\eta,$$

hence

$$\left|\left|u_j(x').y - \alpha_j(x')\right| - \left|P_{N_1}u_j(x_i).y - \alpha_j(x')\right|\right| \leqslant 4\eta\|y\|_{L^2}.$$

Therefore

$$\begin{aligned}\left|\phi(x'+y)\right| &\geqslant \sqrt{\sum_{1\leqslant j\leqslant d'} \left|P_{N_1}u_j(x_i).y - \alpha_j(x')\right|^2} - 4\eta d'\|y\|_{L^2} - C_0\|y\|_{H^\sigma}^2 \\ &\geqslant \sqrt{\sum_{1\leqslant j\leqslant d'} \left|P_{N_1}u_j(x_i).y\right|^2} - 4\eta d'\|y\|_{L^2} - C_0\|y\|_{H^\sigma}^2 - K\varepsilon.\end{aligned}$$

Let W_i be a $(N_1 - d')$-dimensional vector space, orthogonal complementary of V_i in $\mathbb{R}^{N_1}$ for the usual scalar product (L^2 scalar product). Let $e_j = P_{N_1}u_j(x)$, and let us complete $(e_j)_{1\leqslant j\leqslant d'}$ by an orthonormal basis of W_i, to get a basis $(e_j)_{1\leqslant j\leqslant N_1}$ of $\mathbb{R}^{N_1}$. Let

$$Q_1(\beta_1,\ldots,\beta_{d'}) = \sum_{1\leqslant j\leqslant d'} \left|P_{N_1}u_j(x_i).y\right|^2,$$

where

$$y = \sum_{1\leqslant j\leqslant d'} \beta_j e_j,$$

$$Q_2(\beta_1,\ldots,\beta_{d'}) = \left\|\sum_{1\leqslant j\leqslant d'} \beta_j P_{N_1}u_j(x_i)\right\|_{L^2}^2$$

and

$$Q_3(\beta_1,\ldots,\beta_{d'}) = \left\|\sum_{1\leqslant j\leqslant d'} \beta_j P_{N_1}u_j(x_i)\right\|_{H^\sigma}^2.$$

These are three quadratic forms on V_i, which are positive definite. Moreover

$$\left|\phi(x'+y)\right| \geqslant Q_1(\beta)^{1/2} - 4d'\eta Q_2(\beta)^{1/2} - C_0 Q_3(\beta) - K\varepsilon.$$

Let us also introduce

$$Q_0(\beta_1,\dots,\beta_{d'}) = \sum_{1\leqslant j\leqslant d'} \beta_j^2.$$

The quadratic forms Q_0, Q_1, Q_2 and Q_3 are equivalent since we are in a finite-dimensional subspace. In particular, if η is small enough, and C_1 large enough,

$$64{d'}^2\eta^2 Q_2(\beta) < Q_1(\beta) < C_1 Q_2(\beta)$$

and

$$Q_3(\beta) \leqslant C_2 Q_1(\beta)$$

for some constant C_2. The main point is that all these constants are uniform and depend only on the constants given by assumptions (H1) and (H2) and not on x. Note first that (H2) is satisfied for $P_{N_1}u_j(x)$ and $\gamma/2$ instead of γ provided η is small enough.

As the e_j are bounded in L^2, Q_1 is dominated by Q_0 with a constant depending only on d' and M_σ. On the contrary by Gram Schmidt orthogonalization, using (H2), Q_1 is positive definite (with constants depending only on γ, d and M_σ) and dominates Q_0.

Similarly, using (H2), we see that the components β_j of y in the basis e_j are uniformly bounded as $\|y\|_{L^2}$ remains bounded, therefore Q_2 and Q_3 are dominated by Q_0.

Moreover $Q_3 \geqslant Q_2$ and using (H2), Q_2 dominates Q_0.

Hence

$$\left|\phi(x'+y)\right| \geqslant \frac{Q_1(\beta)^{1/2}}{2} - C_0 C_2 Q_1(\beta) - K\varepsilon.$$

Hence if

$$C_0^2 C_2^2 Q_1(\beta) < \frac{1}{16} \tag{9}$$

then

$$\left|\phi(x'+y)\right| \geqslant \frac{Q_1(\beta)^{1/2}}{4} - K\varepsilon. \tag{10}$$

Condition (9) is satisfied if η is small enough (depending on C_0, C_1 and C_2). This gives $|\phi(x'+y)| > \varepsilon$ provided

$$Q_1(\beta)^{1/2} > 4(K+1)\varepsilon$$

which is satisfied provided

$$\|y\|_{L^2} > C_4(K+1)\varepsilon.$$

This leads to

$$\mu_{N_1}\big(A_x \cap \Sigma^{N,\varepsilon}\big) \leqslant C(d')(K+1)^{d'}\varepsilon^{d'}$$

where μ_{N_1} is the Lebesgue measure on $\mathbb{R}^{N_1}$. This is valid for any x, with a constant $C(d')$ independent of x. Summing in x and i ends the proof. $\square$

4. End of the proofs

Let us now apply Section 3 to Section 2, first to quasiresonances then to double eigenvalues.

4.1. *Proof of Theorem 1.2*

In this case

$$\phi = \lambda_j + \lambda_k - \lambda_l.$$

We take $u_1 = d\phi$ defined by

$$u_1(V) = |\psi_j|^2 + |\psi_k|^2 - |\psi_l|^2.$$

Note that assumptions (H1)–(H3) are satisfied, hence Proposition 3.2 implies Theorem 1.2.

4.2. *Proof of Theorem 1.1*

In this case

$$\phi = \lambda_j - \lambda_k.$$

We choose

$$u_1 = |\psi_1|^2 - |\psi_2|^2$$

and

$$u_2 = \psi_1\psi_2.$$

Proposition 3.2 then implies Theorem 1.1.

Remark. Our method being geometric could be used in general context. This explains the general hypothesis (H1)–(H3) with general α_j.

Acknowledgment

The authors would like to thank the referee for his/her remarks and comments.

References

[1] D. Bambusi, Normal forms and semi-classical approximation, Encyclopedia of Mathematical Physics, 2007, in press.

[2] D. Bambusi, B. Grébert, Forme normale pour NLS en dimension quelconque, C. R. Acad. Sci. Paris, Ser. I 337 (2003) 409–414.

[3] V. Bonnaillie, Analyse mathématique de la supraconductivité dans un domaine à coins: méthodes semi-classiques et numériques, thesis, Université Paris 11, 2003.

[4] M. Dauge, B. Helffer, Eigenvalues variation I, Neumann problem for Sturm–Liouville operators, J. Differential Equations 104 (1993) 243–262.

[5] L.H. Elliasson, S.B. Kuksin, KAM for the nonlinear Schrödinger equation, 2007, submitted for publication.

[6] C. Fermanian-Kammerer, P. Gérard, Mesures semi-classiques et croisement de modes, Bull. Soc. Math. France 130 (1) (2002) 123–168.

[7] C. Fermanian-Kammerer, P. Gérard, A Landau Zener formula for non-degenerated involutive codimension 3 crossing, Ann. Inst. Henri Poincaré 4 (2003) 513–552.

[8] B. Helffer, Introduction to the Semiclassical Analysis for the Schrödinger Operator and Applications, Springer-Verlag, 1988.

[9] B. Helffer, J. Sjöstrand, Résonances en limite semi-classique, Mém. Soc. Math. Fr. Sér. 2 (24–25) (1986) 1–228.

[10] D. Henry, Perturbation for the boundary-value problems of partial differential equations, London Math. Soc. Lecture Note Ser. 318 (2006).

[11] P.D. Hislop, J.M. Sigal, Introduction to Spectral Theory with Application to Schrödinger Operators, Appl. Math. Sci., vol. 113, Springer-Verlag, 1991.

[12] T. Kappeler, Multiplicities of the eigenvalues of the Schrödinger equation in any dimension, J. Funct. Anal. 77 (2) (1988) 346–351.

[13] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, Heidelberg, New York, 1986.

[14] C.E. Kenig, D.S. Jerrison, Unique continuation and absence of positive eigenvalues for Schrödinger operator, Ann. of Math. 121 (1985) 463–494.

[15] J. Rauch, Perturbation theory for eigenvalues and resonances of Schrödinger hamiltonians, J. Funct. Anal. 35 (1980) 304–315.

[16] K. Uhlenbeck, Generic properties of eigenfunctions, Amer. J. Math. 98 (1976) 1059–1078.